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On Cyclic Variation-Diminishing Transforms*

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We give a new and more manageable characterization for Cyclic Pólya Frequency functions of order 3 (CPF_3). Our result also improves present knowledge concerning smoothness properties in CPF. In particular, a conjecture of Mairhuber, Schoenberg, and Williamson, On variation-diminishing transformations on the circle, *Rend. Circ. Mat. Palermo* (2) **8** (1959), 1–30, about discontinuous CPF functions is established. © 1994 Academic Press, Inc.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let $\mathscr{A}$ denote the set of functions $f\colon \mathbb{R} \to \mathbb{R}$ with the following properties: f is 2π-periodic and of bounded variation in each period. f is non-negative and satisfies

$$2f(t) = f(t+0) + f(t-0), \qquad t \in \mathbb{R}, \tag{1}$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = 1. \tag{2}$$

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$f \in \mathscr{A}$ is called a Cyclic Pólya Frequency function of order $2r+1$ (written as $f \in \mathrm{CPF}_{2r+1}$) iff

$$\mathrm{Det}\Big(\big(f(x_i - y_j)\big)_{(2k+1,2k+1)}\Big) \geq 0 \tag{3}$$

holds for $k = 0, 1, 2, \ldots, r$, and for any system of real numbers

$$x_1 \leq x_2 \leq \cdots \leq x_{2k+1} < x_1 + 2\pi,$$
$$y_1 \leq y_2 \leq \cdots \leq y_{2k+1} < y_1 + 2\pi.$$

Evidently, $\mathrm{CPF}_3 \supset \mathrm{CPF}_5 \supset \ldots$, and we write

$$\mathrm{CPF} := \bigcap_{r \in \mathbb{N}} \mathrm{CPF}_{2r+1}.$$

This notation was introduced by Karlin [1], but the class CPF was first studied in a 1959 paper by Mairhuber, Schoenberg, and Williamson [3]. In this latter paper the interesting variation diminishing properties of CPF functions were unearthed. To describe this we make use of the following notation. For any finite sequence $(x_1, \ldots, x_n)$ from $\mathbb{R}$ let $v(x_1, \ldots, x_n)$ denote the number of sign changes in that sequence (after deleting all zero terms). For a bounded 2π-periodic function $f\colon \mathbb{R} \to \mathbb{R}$ we define its *cyclic variation* by

$$v_c(f) := \sup v\big(f(x_1), f(x_2), \ldots, f(x_n), f(x_1)\big),$$

where the supremum is extended over all finite selections

$$x_1 < x_2 < \cdots < x_n < x_1 + 2\pi, \qquad n \in \mathbb{N}.$$

THEOREM A. *Let $f \in \mathscr{A}$. Then $f \in \mathrm{CPF}$ iff*

$$v_c(f * g) \leq v_c(g), \tag{4}$$

where

$$(f * g)(t) := \frac{1}{2\pi} \int_0^{2\pi} g(x) f(t-x)\, dx, \qquad t \in \mathbb{R},$$

holds for any piece-wise continuous and 2π-periodic function $g\colon \mathbb{R} \to \mathbb{R}$. More precisely, if $r \in \mathbb{N}$, then $f \in CPF_{2r+1}$ iff (4) *holds for any piece-wise continuous and 2π-periodic function $g\colon \mathbb{R} \to \mathbb{R}$ with $v_c(g) \leq 2r$.*

The CPF part of Theorem A is in [3], while the extended CPF_{2r+1} version has first been published in [1]. For a new, and more direct proof of

Theorem A compare Kurth [2]. The variation-diminishing property discussed in Theorem A is definitely a very interesting subject, and Theorem A seems to be a very elegant solution to the characterization problem for these functions (or kernels). However, the determinant conditions in (3) are not easily dealt with, and they have been of very limited use when it comes to deciding whether a given function has the variation-diminishing property or not. For non-periodic functions (the so-called *line case*) this problem has a longer history, and a complete analytic characterization of the corresponding Pólya Frequency functions in terms of the famous Pólya–Laguerre class of entire functions is available. That such a striking result cannot be expected in the CPF case was already shown in [3]. In fact, nothing much is known even about smoothness properties of CPF or CPF_{2r+1} functions. Karlin [1] has shown that the members of CPF_3 have at most two discontinuities in a period. However, a conjecture made in [3] states that this result is possibly not the best one can obtain, at least for the smaller class CPF. Let, for $\delta > 0$,

$$K_\delta(t) := \begin{cases} Ce^{-t/\delta} & \text{if } 0 < t < 2\pi \\ \frac{1}{2}C(1 + e^{-2\pi/\delta}) & \text{if } t = 0 \end{cases}, \qquad C = \frac{2\pi}{\delta(1 - e^{-2\pi/\delta})}. \tag{5}$$

Then, as shown in [3], K_δ, if extended periodically to $\mathbb{R}$, belongs to CPF.

Conjecture. The functions $K_\delta(\pm t + c)$, where $c \in \mathbb{R}$, are the only non-continuous elements in CPF.

One of the goals of the present paper is to prove this conjecture. In fact, we shall show that the functions mentioned in the conjecture are already the only discontinuous elements in CPF_3 (compare this with Karlin's result mentioned above).

What is actually done in this paper is to give a complete analytic description of the elements in CPF_3. The results obtained here (as far as smoothness properties are concerned) are stronger than what has been known so far even for the much smaller class CPF.

The starting point of our discussion is the obvious fact that every CPF_3 function (via Theorem A) preserves periodic monotonicity. A 2π-periodic, piecewise continuous function g which satisfies (1) is called *periodically monotone* ($g \in \mathrm{PM}$) iff

$$\forall c \in \mathbb{R}\colon v_c(g + c) \leq 2. \tag{6}$$

A function $f \in \mathscr{A}$ is said to preserve periodic monotonicity ($f \in$ PMP) iff

$$\forall g \in \mathrm{PM}: f * g \in \mathrm{PM}.$$

The class PMP (without the—for our present purpose useful—restriction: $f \in \mathscr{A}$) has been studied and completely characterized in [4].

THEOREM B. *Let $f \in \mathscr{A}$. Then $f \in$ PMP iff it satisfies (a)–(c) below.*

(a) *f is periodically monotone.*

(b) *f has at most two discontinuities in a period, and in each such discontinuity s we have* $|f(s+0) - f(s-0)| = \sup_{\mathbb{R}} f - \inf_{\mathbb{R}} f$.

(c) *f is continuously differentiable in any open interval in which f assumes neither its supremum nor its infimum. Furthermore,* $\log|f'|$ *is concave in those intervals.*

Since $\mathrm{CPF}_3 \subset$ PMP, the necessary conditions expressed in Theorem B are valid for any $f \in \mathrm{CPF}_3$ as well.

In the sequel we shall use the following abbreviation:

$$D_f\begin{pmatrix} x_1, x_2, x_3 \\ y_1, y_2, y_3 \end{pmatrix} := \mathrm{Det}\big((f(x_i - y_j))_{(3,3)}\big).$$

Assume for the moment that a function $f \in \mathrm{CPF}_3$ belongs to $C^4(\mathbb{R})$. A simple verification shows then

$$\begin{aligned} \lim_{h \to 0^+} h^{-6} D_f\begin{pmatrix} x+h & x+2h & x+3h \\ h & 2h & 3h \end{pmatrix} &= \begin{vmatrix} f''(x) & f'(x) & f(x) \\ f'''(x) & f''(x) & f'(x) \\ f^{(4)}(x) & f'''(x) & f''(x) \end{vmatrix} \\ &= \frac{-F^2(x)}{f(x)} \frac{d^2 \log(F(x))}{dx^2} \\ &\geq 0, \end{aligned} \tag{7}$$

for $x \in \mathbb{R}$, where

$$F(x) := \left| f''(x) f(x) - (f'(x))^2 \right|.$$

This implies that in each interval where both, f and F, are positive we have

$$\log\left| f''(x) f(x) - (f'(x))^2 \right| \text{ concave,} \tag{8}$$

a fact which does not use the existence of the third and fourth derivative of f. This is an important observation and actually the guide to our main results, which we are now ready to state explicitly. If I is some open interval of $\mathbb{R}$ then we shall write $f \in L(I)$ if $f \in C^2(I)$, f and $|f''(x)f(x) - (f'(x))^2|$ are positive in I, and f satisfies (8) in I.

THEOREM 1. *The only functions in* CPF_3 *which are not in* $C^0(\mathbb{R})$ *are those mentioned in the conjecture above.*

THEOREM 2. *Let* $f \in \mathscr{A} \cap C^0(\mathbb{R})$ *and assume that* f *has zeros. Then* f *belongs to* CPF_3 *if and only if it has the following structure*:

(a) *There exist points* $x_a \le x_b < x_a + 2\pi$ *such that* $f \equiv 0$ *in* $[x_a, x_b]$.

(b) *For* $I := (x_b, x_a + 2\pi)$ *we have* $f \in L(I)$.

THEOREM 3. *Let* $f \in \mathscr{A} \cap C^0(\mathbb{R})$ *be zero-free. Then* $f \in \mathrm{CPF}_3$ *if and only if it satisfies the conditions* (a)–(c) *below.*

(a) *The one-sided derivatives* f'_+, f'_- *of* f *exist everywhere and the function*

$$P_f(x) := (f'_+(x) + f'_-(x))/(2f(x)) \tag{9}$$

is periodically monotone.

(b) P_f *has at most two discontinuities in a period, and at each such discontinuity* s *we have* $|P_f(s+0) - P_f(s-0)| = \sup_{\mathbb{R}} P_f - \inf_{\mathbb{R}} P_f$.

(c) *In each open interval* I, *in which* P_f *assumes neither its supremum nor its infimum, we have* $f \in L(I)$.

Note that a CPF_3 function is continuously differentiable with the exception of at most 2 points in a period and is twice continuously differentiable with the exception of at most 4 points in a period. It is worthwhile to remark that P_f, if f has two points in a period where it is not differentiable, is a very simple step-function, so that f is indeed C^∞, except at those two points (where f takes its maximum and minimum). It is, of course, easy to write down these functions explicitly.

Theorem 3 has another interesting consequence which we mention in passing (a somewhat weaker version of it will be instrumental in the proof of Theorem 3).

THEOREM 4. *Assume that* $f \in \mathscr{A} \cap C^2(\mathbb{R})$ *is zero-free. Then* $f \in \mathrm{CPF}_3$ *if and only if*

$$\frac{f''(t)}{f(t)} + i\frac{f'(t)}{f(t)}, \qquad 0 \le t < 2\pi, \tag{10}$$

is a Jordan curve with a convex interior domain.

As has been pointed out before, the "only if" parts of Theorems 1–4 apply to CPF as well. Since our results come only from the CPF_3 class, it is very likely that in CPF much higher smoothness prevails. At this point, however, it is not easy to make a reasonable guess in this direction.

The plan of this paper is to prove Theorem 1 in Section 2, to establish the "only if" parts of Theorems 2, 3 in Section 3, and the other directions in Section 4.

2. Proof of Theorem 1

We prepare this proof with some lemmas. These will be useful also in later sections.

LEMMA 1. *A function $f \in \mathrm{CPF}_3$ has at most one discontinuity in a period.*

Proof. From Theorem B we know that f has at most two discontinuities in a period, and if it has two, then it must be a step function. Without loss of generality assume that $0, 2s \in [0, 2\pi)$ are the points of discontinuity of f, which then, in $[0, 2\pi)$, may be written

$$f(x) = \begin{cases} M & \text{for } x \in (0, 2s) \\ \frac{1}{2}(m + M) & \text{for } x = 0, x = 2s \\ m & \text{for } x \in (2s, 2\pi), \end{cases}$$

where $0 \le m < M$. Choosing $\varepsilon := \min\{(\pi - s)/4, s/4\}$ one obtains

$$D_f\begin{pmatrix} 3\varepsilon & s + 2\varepsilon & 2s + \varepsilon \\ 0 & 2\varepsilon & 4\varepsilon \end{pmatrix} = \begin{vmatrix} M & M & m \\ M & M & M \\ m & M & M \end{vmatrix} = -M(M - m)^2 < 0,$$

which contradicts $f \in \mathrm{CPF}_3$. ∎

LEMMA 2. *Let $f \in \mathrm{CPF}_3$ and $M := \sup_{\mathbb{R}} f > m := \inf_{\mathbb{R}} f$. Then, in any given period, f assumes M in at most one point. The same holds for m if $m > 0$.*

Proof. If f assumes M at more than one point in a period, then we deduce from Theorem B that f assumes M on an interval of length $2s \in (0, 2\pi)$ (it is understood that $2s$ is the largest possible of those numbers), and Lemma 1 tells us that f is continuous on at least one of the endpoints of this interval. In fact, we can normalize the situation so that f is continuous at 0, and $f(t) = M$ for $0 \le t < 2s$. We now may choose

$\varepsilon \in (0, s/2)$ so that f is strictly increasing in $(-5\varepsilon, 0)$. Then

$$D_f\begin{pmatrix} -3\varepsilon & s & 2s-\varepsilon \\ -2\varepsilon & 0 & \varepsilon \end{pmatrix} = \begin{vmatrix} f(-\varepsilon) & f(-3\varepsilon) & f(-4\varepsilon) \\ f(s+2\varepsilon) & f(s) & f(s-\varepsilon) \\ f(2s+\varepsilon) & f(2s-\varepsilon) & f(2s-2\varepsilon) \end{vmatrix}$$

$$= \begin{vmatrix} f(-\varepsilon) & f(-3\varepsilon) & f(-4\varepsilon) \\ M & M & M \\ f(2s+\varepsilon) & M & M \end{vmatrix}$$

$$= M(M - f(2s+\varepsilon))(f(-4\varepsilon) - f(-3\varepsilon))$$

$$< 0.$$

For ε sufficiently small this contradicts $f \in \mathrm{CPF}_3$. A similar argument works for m, but obviously only if $m > 0$. ∎

The following simple observation will be useful.

LEMMA 3. *Let I be some open interval and $f \in C^1(I)$ be positive with f'/f strictly monotone in I. Then, for $t_1, t_2 \in I$, $t_1 < t_2$ the function*

$$F(h) := \begin{vmatrix} f(t_1+h) & f(t_1) \\ f(t_2+h) & f(t_2) \end{vmatrix}$$

is positive for small $h > 0$ if f'/f increases and negative if f'/f decreases.

For a proof of this lemma just note that $F'(0)$ is positive if f'/f increases and negative if it decreases.

Proof of Theorem 1. Let $f \in \mathrm{CPF}_3$ be not continuous. Without loss of generality we may assume that $s = 0$ is such a discontinuity of f, and that $f(0+0) > f(0-0)$. If $f(0-0) = 0$ then let $(0, \theta) \subset (0, 2\pi)$ be the largest interval in which f is positive, in all other cases set $\theta = 2\pi$. Using Lemma 2 and $f \in \mathrm{PMP}$ we deduce that f is strictly decreasing and continuously differentiable in $I := (0, \theta)$.

We wish to show that $f'/f = \text{const.}$ in I. Assume there exists an interval $I_1 \subset I$ on which f'/f is strictly decreasing. Then we choose $t_1, t_2 \in I_1$, $t_1 < t_2$. For small positive numbers ε, h we have

$$t_1 - \varepsilon < t_1 + \varepsilon < t_2 + \varepsilon < t_1 - \varepsilon + 2\pi \quad \text{and} \quad -h < 0 < t_1 < -h + 2\pi.$$

Now

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} D_f\begin{pmatrix} t_1-\varepsilon & t_1+\varepsilon & t_2+\varepsilon \\ -h & 0 & t_1 \end{pmatrix} \\
&= \lim_{\varepsilon\to 0^+}\begin{vmatrix} f(t_1-\varepsilon+h) & f(t_1-\varepsilon) & f(-\varepsilon) \\ f(t_1+\varepsilon+h) & f(t_1+\varepsilon) & f(\varepsilon) \\ f(t_1+\varepsilon+h) & f(t_1+\varepsilon) & f(t_2-t_1+\varepsilon) \end{vmatrix} \\
&= \begin{vmatrix} f(t_1+h) & f(t_1) & f(0-0) \\ f(t_1+h) & f(t_1) & f(0+0) \\ f(t_2+h) & f(t_2) & f(t_2-t_1) \end{vmatrix} \\
&= -(f(0+0)-f(0-0))\begin{vmatrix} f(t_1+h) & f(t_1) \\ f(t_2+h) & f(t_2) \end{vmatrix} \\
&< 0,
\end{aligned}$$

where we made use of Lemma 3, applied to I_1. This contradicts $f \in \mathrm{CPF}_3$. Similar reasoning yields

$$\lim_{\varepsilon\to 0^+} D_f\begin{pmatrix} t_1 & t_2-\varepsilon & t_2+\varepsilon \\ -h & 0 & t_2 \end{pmatrix} < 0$$

under the assumption that f'/f is strictly decreasing in I_1.

We can now conclude that $f(t) = Ce^{\lambda t}$ for $t \in (0, \theta)$ and some constants $C > 0$, $\lambda < 0$, which *per se* implies that $\theta = 2\pi$. This completes the proof of Theorem 1, taking into account the normalizations we had applied to f. ∎

3. Sufficient Conditions for CPF_3

A crucial tool in our work is connected with convex curves in the complex plane. They are defined as follows. A piecewise continuous bounded function $\Gamma(t) = \gamma_1(t) + i\gamma_2(t)$: $[0, 2\pi) \to \mathbb{C}$ is called a convex curve if

$$\forall a, b, c \in \mathbb{R}: v_c(a + b\gamma_1 + c\gamma_2) \leq 2, \tag{11}$$

where γ_1, γ_2 are assumed to be periodically extended to $\mathbb{R}$. If, for instance, Γ happens to be a Jordan curve, then this condition means that the interior domain of this curve is a convex set, which accounts for the name "convex curve." However, it is easy to obtain some geometric feeling for "convex curves" also in the more general cases. Another, equivalent description of the condition (11) is that

$$\begin{vmatrix} 1 & \gamma_1(x_1) & \gamma_2(x_1) \\ 1 & \gamma_1(x_2) & \gamma_2(x_2) \\ 1 & \gamma_1(x_3) & \gamma_2(x_3) \end{vmatrix} \tag{12}$$

is of constant sign for all choices of $x_1 < x_2 < x_3 < x_1 + 2\pi$. If this sign is positive, then we say that Γ is positively convex. It can be seen that this means that the boundary of the closed convex hull of $\{\Gamma(t)\colon 0 \le t < 2\pi\}$, is traversed in the positive direction by $\Gamma(t)$. If the determinants (12) are all strictly positive, then $\Gamma(t)$ is called a strictly positive convex curve. Geometrically this means that Γ, in addition to the other already mentioned properties, maps no interval onto a line segment or a single point.

LEMMA 4. *Assume that $f \in \mathscr{A}$ is a zero-free, non-constant trigonometric polynomial. Then $f \in \mathrm{CPF}_3$ if and only if*

$$\frac{f''(t)}{f(t)} + i\frac{f'(t)}{f(t)}, \qquad 0 \le t < 2\pi, \tag{13}$$

is a strictly positive convex curve.

Proof. Assume $f \in \mathrm{CPF}_3$, so that

$$\lim_{h \to 0^+} h^{-3} D_f\begin{pmatrix} x_1 & x_2 & x_3 \\ -h & 0 & h \end{pmatrix} = \begin{vmatrix} f''(x_1) & f'(x_1) & f(x_1) \\ f''(x_2) & f'(x_2) & f(x_2) \\ f''(x_3) & f'(x_3) & f(x_3) \end{vmatrix} \ge 0 \tag{14}$$

for all choices of $x_1 < x_2 < x_3 < x_1 + 2\pi$. Hence

$$\begin{vmatrix} 1 & f''(x_1)/f(x_1) & f'(x_1)/f(x_1) \\ 1 & f''(x_2)/f(x_2) & f'(x_2)/f(x_2) \\ 1 & f''(x_3)/f(x_3) & f'(x_3)/f(x_3) \end{vmatrix} \ge 0, \tag{15}$$

which shows that (13) defines a positively convex curve. Since f is a trigonometric polynomial it is easily seen that this curve cannot contain line elements and is never constant on intervals. Therefore it is strictly convex.

Now assume that the curve (13) is strictly positively convex so that the determinants (15) are all positive. Assume there exist numbers

$$x_1 < x_2 < x_3 < x_1 + 2\pi, \qquad y_1 < y_2 < y_3 < y_1 + 2\pi$$

with

$$D_f\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} = \begin{vmatrix} f(x_1 - y_1) & f(x_1 - y_2) & f(x_1 - y_3) \\ f(x_2 - y_1) & f(x_2 - y_2) & f(x_2 - y_3) \\ f(x_3 - y_1) & f(x_3 - y_2) & f(x_3 - y_3) \end{vmatrix} < 0. \quad (16)$$

We fix the numbers x_j, $j = 1, 2, 3$, and distinguish two cases, namely whether (16), with < 0 replaced by ≤ 0 holds for *all* choices of $y_1 < y_2 < y_3 < y_1 + 2\pi$ or not. In the first case we see as in (14)

$$\frac{1}{f(x_1)f(x_2)f(x_3)} \lim_{h \to 0^+} h^{-3} D_f\begin{pmatrix} x_1 & x_2 & x_3 \\ -h & 0 & h \end{pmatrix}$$
$$= \begin{vmatrix} 1 & f''(x_1)/f(x_1) & f'(x_1)/f(x_1) \\ 1 & f''(x_2)/f(x_2) & f'(x_2)/f(x_2) \\ 1 & f''(x_3)/f(x_3) & f'(x_3)/f(x_3) \end{vmatrix} \leq 0,$$

which contradicts our assumption on f.

In the other case it is evident that the closed curve

$$\Gamma(t) := \frac{f(x_2 - t)}{f(x_1 - t)} + i\frac{f(x_3 - t)}{f(x_1 - t)}, \qquad 0 \leq t < 2\pi,$$

is not convex. There are two possible (not mutually exclusive) reasons for that: either (a) the tangent vector does not turn monotonically, or (b) the curve has self-intersections.

(a) In this case

$$\left(\frac{f(x_3 - t)}{f(x_1 - t)}\right)''\left(\frac{f(x_2 - t)}{f(x_1 - t)}\right)' - \left(\frac{f(x_3 - t)}{f(x_1 - t)}\right)'\left(\frac{f(x_2 - t)}{f(x_1 - t)}\right)''$$
$$= -\frac{f(x_2 - t)f(x_3 - t)}{f(x_1 - t)^2}$$
$$\times \begin{vmatrix} 1 & f''(x_3 - t)/f(x_3 - t) & f'(x_3 - t)/f(x_3 - t) \\ 1 & f''(x_2 - t)/f(x_2 - t) & f'(x_2 - t)/f(x_2 - t) \\ 1 & f''(x_1 - t)/f(x_1 - t) & f'(x_1 - t)/f(x_1 - t) \end{vmatrix}$$

takes positive and negative values (with t varying). As above, this contradicts the assumption on f.

(b) The support of Γ is completely contained in the upper right quadrant of the complex plane. If Γ has self-intersections this means that $(d/dt)\arg \Gamma(t)$ changes sign at least four times in a period. But

$$\begin{aligned}\frac{d}{dt}\arg \Gamma(t) &= \frac{d}{dt}\arctan\frac{f(x_3-t)}{f(x_2-t)}\\ &= \frac{f(x_2-t)}{(f(x_2-t))^2+(f(x_3-t))^2}\frac{1}{f(x_3-t)}\\ &\quad\times\left(\frac{f'(x_3-t)}{f(x_3-t)}-\frac{f'(x_2-t)}{f(x_2-t)}\right),\end{aligned}$$

so that

$$v_c\left(\frac{f'(x_3-t)}{f(x_3-t)}-\frac{f'(x_2-t)}{f(x_2-t)}\right)\geq 4. \tag{17}$$

Using (11) and (6), the convexity of (13) shows that f'/f is periodically monotone, and this obviously contradicts (17). ∎

It is well-known (cf. [1]) that the de-la-Vallée–Poussin kernels

$$V_n(x) := \binom{2n}{n}^{-1}(\cos(x/2))^{2n}, \qquad n\in\mathbb{N},$$

are in CPF and therefore cyclic variation-diminishing. Furthermore, if g is a 2π-periodic piecewise continuous bounded function satisfying (1) then

$$\lim_{n\to\infty}(V_n * g)(x) = \tfrac{1}{2}(g(x+0)+g(x-0)), \qquad x\in\mathbb{R}. \tag{18}$$

If g is continuous in a compact interval, then the convergence (18) is uniform in that interval.

These properties lead easily to the following conclusion:

$$f\in \mathrm{CPF}_3 \Leftrightarrow \forall n\in\mathbb{N}\colon V_n * f\in\mathrm{CPF}_3. \tag{19}$$

Note that Lemma 4 is a somewhat weaker version of Theorem 4, which, in turn, can be seen as a sort of limiting case of Lemma 4, using the V_n. We omit the details.

The next lemma establishes the link between convex curves as described above, and the class $L(I)$ (compare (8)).

LEMMA 5. *Assume that* $f \in C^2((a,b))$ *with* $f > 0$, *and* $(f'/f)' > 0$ (< 0) *in* (a,b). *Then we have* $f \in L((a,b))$, *i.e.*,

$$\log\left| f''(x)f(x) - (f'(x))^2 \right| \text{ concave in } (a,b),$$

iff there exists a concave (convex) function $H\colon (f'(a)/f(a), f'(b)/f(b)) \to \mathbb{R}$ *satisfying*

$$\frac{f''(x)}{f(x)} = H\left(\frac{f'(x)}{f(x)}\right), \qquad x \in (a,b).$$

Proof. Note that under our general assumptions the function H exists and is continuous. Following the idea in [4, Lemma 12] we only need to show that for $x \in (a,b)$ the relation

$$\lim_{\varepsilon \to 0^+} \frac{F(x+\varepsilon) - F(x)}{\varepsilon} = \lim_{\varepsilon \to 0^+} \frac{H(y+\varepsilon) - H(y)}{\varepsilon}\bigg|_{y = f'(x)/f(x)} \tag{20}$$

with

$$F(x) = \log\left| f''(x)f(x) - (f'(x))^2 \right|$$

holds if at least one of the two limits involved exists. And the corresponding relation should be true for the left-sided limits as well.

Assume that the limit on the left of (20) exists and equals α. We then have

$$\begin{aligned}
\alpha &= \lim_{\varepsilon \to 0^+} \frac{\log((f'/f)')|_{x+\varepsilon} - \log((f'/f)')|_x}{\varepsilon} + 2\log(f(x))' \\
&= \lim_{\varepsilon \to 0^+} \frac{\log\left(H(f'/f) - (f'/f)^2\right)\big|_{x+\varepsilon} - \log\left(H(f'/f) - (f'/f)^2\right)\big|_x}{\varepsilon} \\
&\quad + 2\log(f(x))'
\end{aligned}$$

Applying the mean-value theorem to the log-function we can then write (using $\lim_{\varepsilon \to 0^+} \theta(\varepsilon) = 0$)

$$\begin{aligned}
\alpha &= \lim_{\varepsilon \to 0^+} \frac{\left(H(f'/f) - (f'/f)^2\right)\big|_{x+\varepsilon} - \left(H(f'/f) - (f'/f)^2\right)\big|_x}{\varepsilon\left[\left(H(f'/f) - (f'/f)^2\right)\big|_x + \theta(\varepsilon)\right]} \\
&= \lim_{\varepsilon \to 0^+} \frac{H(f'/f)|_{x+\varepsilon} - H(f'/f)|_x}{\varepsilon (f'/f)'|_x}.
\end{aligned}$$

We have

$$\left.\frac{f'}{f}\right|_{x+\varepsilon} = \left.\frac{f'}{f}\right|_{x} + \varepsilon\left.\left(\frac{f'}{f}\right)'\right|_{x+\delta(\varepsilon)}$$

with $\delta(\varepsilon)$ positive for positive (small) ε, and continuously extendable into $\varepsilon = 0$ with $\delta(0) = 0$. It is clear that these properties of δ are shared by the function

$$\gamma(\varepsilon) := \varepsilon\left.\left(\frac{f'}{f}\right)'\right|_{x+\delta(\varepsilon)},$$

which eventually leads to

$$\begin{aligned}\alpha &= \lim_{\varepsilon\to 0^+} \frac{H((f'/f)|_x + \gamma(\varepsilon)) - H((f'/f)|_x)}{\gamma(\varepsilon)} \frac{(f'/f)'|_{x+\delta(\varepsilon)}}{(f'/f)'|_x} \\ &= \lim_{\varepsilon\to 0^+} \left.\frac{H(y+\varepsilon) - H(y)}{\varepsilon}\right|_{y=f'(x)/f(x)},\end{aligned}$$

the assertion. ∎

We now begin with the proof of the "if" part of Theorem 3. We first study the case where P_f is continuous in $\mathbb{R}$, and define

$$c_1 := \min_{\mathbb{R}} P_f, \qquad c_2 := \max_{\mathbb{R}} P_f.$$

We assume that these values are taken by P_f in $I_b := [x_1, x_2]$ and in $I_t := [x_3, x_4]$, respectively, where

$$x_1 \le x_2 < x_3 \le x_4 < x_1 + 2\pi,$$

but nowhere else. We set

$$I_i := (x_2, x_3), \qquad I_d := (x_4, x_1 + 2\pi),$$

so that P_f strictly increases in I_i, and strictly decreases in I_d. The assumption in Theorem 3 now is that $f \in L(I_i) \cap L(I_d)$. Lemma 5 then yields a concave function H_1 and a convex function H_2 with

$$\frac{f''(x)}{f(x)} = \begin{cases} H_1(f'(x)/f(x)), & x \in I_i, \\ H_2(f'(x)/f(x)), & x \in I_d. \end{cases}$$

We note that this implies the existence of the left-hand and right-hand limits of f'' everywhere, and we define

$$f_2(x) := \tfrac{1}{2}(f''(x-0) + f''(x+0)), \qquad x \in \mathbb{R}.$$

We study the behavior of the complex function

$$\Omega(x) := \frac{f'(x)}{f(x)} + i\frac{f_2(x)}{f(x)} \tag{21}$$

for $x \in I_i \cup I_d$. In I_i we have $(f'/f)' > 0$, and hence

$$\frac{f''(x)}{f(x)} > \left(\frac{f'(x)}{f(x)}\right)^2, \qquad x \in I_i.$$

This shows that if $(\mathrm{Re}(\Omega), \mathrm{Im}(\Omega))$ is looked at in a (u, v) plane, its restriction to I_i is a concave curve above the parabola (u, u^2), $u \in \mathbb{R}$. A similar observation for I_d shows that Ω restricted to I_d is a convex curve below that same parabola. The projection to the u-axis is in both cases the interval $c_1 < u < c_2$. Also note that Ω is a 1–1-mapping in $I_i \cup I_d$. In the remaining intervals (which may degenerate to points) Ω is piecewise stationary, with the real part constantly equal to c_1 or c_2, and the imaginary part monotonically moving. It is now clear that Ω is a "convex curve" in the sense discussed earlier in this section. In fact it is readily seen that it is negatively oriented. We now make use of (11) to conclude that

$$\forall a, b, c \in \mathbb{R}: v_c\big(af(x) + bf'(x) + cf_2(x)\big) \leq 2$$

Using the CPF property of the de-la-Vallée-Poussin kernels this implies

$$\forall a, b, c \in \mathbb{R}: v_c\big(a(V_n * f)(x) + b(V_n * f')(x) + c(V_n * f_2)(x)\big) \leq 2$$

holds for $n \in \mathbb{N}$. Furthermore, $(V_n * f)(x) > 0$ for all x, and therefore we may deduce that

$$\Omega_n(x) := \frac{(V_n * f')(x)}{(V_n * f)(x)} + i\frac{(V_n * f_2)(x)}{(V_n * f)(x)}$$

is a convex curve as well, and at least for large n it will be negatively oriented. Using integration by parts one easily verifies that

$$V_n * f' = (V_n * f)' \qquad \text{and} \qquad V_n * f_2 = (V_n * f)'',$$

and this implies that the curves

$$\frac{(V_n * f)''}{V_n * f} + i\frac{(V_n * f)'}{V_n * f}$$

are strictly positive convex curves (note that $V_n * f$ is a trigonometric polynomial). Lemma 4 now proves that $V_n * f$ is in CPF_3 for all n large enough. The defining property for CPF_3 and (18) completes the proof of Theorem 3 for continuous P_f.

The "if" parts of Theorem 2 and the remaining cases of Theorem 3 will be reduced to the case just discussed, showing that the functions in question are only limiting cases of those already studied. This is based on a suitable "patch" function which we are going to discuss in the next lemma.

LEMMA 6. *Let* $\alpha, \beta, \gamma, x_0, x_1 \in \mathbb{R}$ *satisfy* $\alpha\beta < 0, \gamma > 0, x_0 < x_1 < x_0 + 2\pi$. *Then there exists* $F \in L((x_0, x_1))$ *with*

$$F(x_0) = F(x_1) = \gamma, \tag{22}$$

$$\left.\frac{F'}{F}\right|_{x=x_0} = \alpha, \qquad \left.\frac{F'}{F}\right|_{x=x_1} = \beta, \tag{23}$$

where the derivative is with respect to the variable x.

If necessary we shall write $F(\alpha, \beta, \gamma, x_0, x_1; x)$ instead of $F(x)$.

Proof. It can be shown that there is a function of the form

$$F(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}, \qquad C_1 C_2 \neq 0,\ \lambda_1 \neq \lambda_2, \tag{24}$$

which satisfies the four conditions (22), (23). Functions of the form (24) have at most one (simple) zero (if they do not vanish identically), and this together with (22) implies $F > 0$ in the interval (x_0, x_1). A calculation yields

$$F''(x)F(x) - (F'(x))^2 = C_1 C_2 (\lambda_1 - \lambda_2)^2 e^{(\lambda_1 + \lambda_2)x},$$

and this shows $F \in L((x_0, x_1))$. ∎

We continue with the proof of Theorem 3, and assume now that P_f has exactly one discontinuity, which in the notation given above means that either I_i or I_d has collapsed. Assume I_d has collapsed.

It is easily seen that the discontinuity of P_f, namely at $x_4 = x_1 + 2\pi$, corresponds to the (only) point where f assumes its maximal value in the period. In fact, f is continuous, non-constant, and differentiable except in

x_4, by assumption. This means that P_f assumes positive and negative values, i.e., $c_1 < 0, c_2 > 0$. Hence f increases in $(x_4 - \varepsilon, x_4)$ and decreases in $(x_4, x_4 + \varepsilon)$, for some small $\varepsilon > 0$. Since the assumptions on P_f also imply that f is periodically monotone, the claim follows.

Let $\varepsilon > 0$ be small. We can find $x_4(\varepsilon) < x_4$ and $x_1(\varepsilon) > x_1$ so that

$$f(x_4(\varepsilon)) = f(x_1(\varepsilon) + 2\pi) = f(x_4) - \varepsilon.$$

From Lemma 6 it now follows that the 2π-periodic function f_ε, whose restriction to the period $(x_1(\varepsilon), x_1(\varepsilon) + 2\pi)$ is defined by

$$f_\varepsilon(x) := \begin{cases} f(x) & x_1(\varepsilon) \le x \le x_4(\varepsilon) \\ F(P_f(x_4(\varepsilon)), P_f(x_1(\varepsilon) + 2\pi), f(x_4) - \varepsilon, & \\ \quad x_4(\varepsilon), x_1(\varepsilon) + 2\pi; x), & x_4(\varepsilon) < x < x_1(\varepsilon) + 2\pi \end{cases},$$

has P_{f_ε} continuous and satisfies (a)–(c) of Theorem 3. The same is true for $\tilde{f}_\varepsilon := c_\varepsilon f_\varepsilon$, where c_ε is the appropriate positive constant to make $\tilde{f}_\varepsilon$, satisfy the normalization (2). Hence, by the part of Theorem 3 already proved, we conclude that $\tilde{f}_\varepsilon \in \mathrm{CPF}_3$. Letting $\varepsilon \to 0$ we find $f \in \mathrm{CPF}_3$.

If P_f has two discontinuities, a similar discussion as above (with two patches) can be applied to again obtain $f \in \mathrm{CPF}_3$. This completes the proof of the "if"-part of Theorem 3.

Next assume that f satisfies the assumptions of Theorem 2. Then a similar construction as used just before (using a positive patch for $(x_2(\varepsilon), x_1(\varepsilon) + 2\pi)$, where $x_1(\varepsilon) < x_a, x_2(\varepsilon) > x_b$ are chosen so that $f(x_1(\varepsilon)) = f(x_2(\varepsilon)) = \varepsilon$) leads to a function $\tilde{f}_\varepsilon$ which satisfies the assumptions of Theorem 3, and hence $\tilde{f}_\varepsilon \in \mathrm{CPF}_3$. Taking the limit $\varepsilon \to 0$ completes the proof of the "if"-part of Theorem 2.

4. Necessary Conditions for CPF_3

In this section we shall prove the "only if" parts of Theorems 2, 3. It is therefore generally assumed that $f \in \mathrm{CPF}_3 \cup C^0(\mathbb{R})$, and f non-constant.

We find a unique $x_M \in [0, 2\pi)$ with $f(x_M) = \max_{\mathbb{R}} f$ (Lemma 1). Furthermore, f, as a member of PMP, is periodically monotone and non-negative (Theorem B). Thus we find $x_a, x_b \in (x_M, x_M + 2\pi)$, $x_a \le x_b$ with $f(x) = \min_{\mathbb{R}} f$ for $x_a \le x \le x_b$ and $f(x) > \min_{\mathbb{R}} f$ for $x_b < x < x_a + 2\pi$. Lemma 1 implies that $x_a = x_b$ whenever f is zero-free. We set $I_1 = (x_M, x_a)$, $I_2 = (x_b, x_M + 2\pi)$.

We begin with the statement of some immediate consequences of Theorem B for our f.

LEMMA 7. *Let f be as above. Then the following statements hold true.*

(a) *The one-sided derivatives f'_+, f'_- exist everywhere and are bounded. $\frac{1}{2}(f'_+ + f'_-)$ is periodically monotone.*

(b) *f'/f is continuous and non-vanishing in $I_1 \cup I_2$.*

(c) *The one-sided derivatives $f''_\pm := (f')'_\pm$ exist in $I_1 \cup I_2$, and $f''_\pm/f'$ are decreasing functions there. f'' exists almost everywhere.*

The actual behavior of f in $I_1 \cup I_2$ is now derived in a series of lemmas.

LEMMA 8. *Assume there exists an open interval $I \subset I_1 \cup I_2$ on which f'/f strictly decreases. Then f is continuously differentiable in x_M. Similarly, if $x_a = x_b$ and f'/f is strictly increasing on I, then f is continuously differentiable at x_a.*

Proof. Assume that f'/f strictly decreases in I, and choose $t_1, t_2 \in I$, $t_1 < t_2$. Using the fact that $f'_-(x_M) \geq 0 \geq f'_+(x_M)$ we obtain

$$\lim_{h\to 0^+} h^{-2} D_f\begin{pmatrix} x_m & t_1 & t_2 \\ -h & 0 & h \end{pmatrix}$$

$$= \lim_{h\to 0^+} h^{-2}\begin{vmatrix} f(x_m+h) & f(x_m) & f(x_m-h) \\ f(t_1+h) & f(t_1) & f(t_1-h) \\ f(t_2+h) & f(t_2) & f(t_2-h) \end{vmatrix}$$

$$= \lim_{h\to 0^+}\begin{vmatrix} (f(x_M+h)-f(x_M))/h & (f(x_M)-f(x_M-h))/h & f(x_M) \\ (f(t_1+h)-f(t_1))/h & (f(t_1)-f(t_1-h))/h & f(t_1) \\ (f(t_2+h)-f(t_2))/h & (f(t_2)-f(t_2-h))/h & f(t_2) \end{vmatrix}$$

$$= \begin{vmatrix} f'_+(x_M) & f'_-(x_M) & f(x_M) \\ f'(t_1) & f'(t_1) & f(t_1) \\ f'(t_2) & f'(t_2) & f(t_2) \end{vmatrix}$$

$$= -f(t_1)f(t_2)(f'_-(x_M) - f'_+(x_M))(f'(t_1)/f(t_1) - f'(t_2)/f(t_2))$$

$$\leq 0.$$

For $f \in \mathrm{CPF}_3$ this is only possible if equality holds, which implies $f'_-(x_M) = f'_+(x_M) = 0$, and therefore, using Lemma 7(a), (b), the assertion. The other case can be established in a similar fashion. ∎

LEMMA 9. *Let I be some interval in which f is differentiable and positive, and where P_f assumes neither $\sup_{\mathbb{R}} P_f$ nor $\inf_{\mathbb{R}} P_f$. Then P_f is strictly monotone in I.*

Proof. Assume there are points $x_1 < x_2$ in I with $P_f(x_1) = P_f(x_2)$. Choose a $y \in (0, 2\pi)$ with

$$\frac{f(x_1 - y)}{f(x_1)} \neq \frac{f(x_2 - y)}{f(x_2)}.$$

Since P_f assumes both larger and smaller values than $P_f(x_1)$ in $(x_2, x_1 + 2\pi)$, we can find x_3 in that latter interval so that

$$\operatorname{sgn}\big(P_f(x_2) - P_f(x_3)\big) = -\operatorname{sgn}\left(\frac{f(x_1 - y)}{f(x_1)} - \frac{f(x_2 - y)}{f(x_2)}\right).$$

Then

$$\begin{aligned}
&\lim_{h \to 0^+} h^{-1} D_f\begin{pmatrix} x_1 & x_2 & x_3 \\ -h & 0 & y \end{pmatrix} \\
&\quad = \begin{vmatrix} f'(x_1) & f(x_1) & f(x_1 - y) \\ f'(x_2) & f(x_2) & f(x_2 - y) \\ f'(x_3) & f(x_3) & f(x_3 - y) \end{vmatrix} \\
&\quad = f(x_1) f(x_2) f(x_3) \begin{vmatrix} P_f(x_2) & 1 & f(x_1 - y)/f(x_1) \\ P_f(x_2) & 1 & f(x_2 - y)/f(x_2) \\ P_f(x_3) & 1 & f(x_3 - y)/f(x_3) \end{vmatrix} \\
&\quad = f(x_1) f(x_2) f(x_3) \left(\frac{f(x_1 - y)}{f(x_1)} - \frac{f(x_2 - y)}{f(x_2)}\right)\big(P_f(x_2) - P_f(x_3)\big) \\
&\quad < 0,
\end{aligned}$$

which is impossible for $f \in \mathrm{CPF}_3$. ∎

LEMMA 10. *Assume that f has zeros. Then f'/f is continuous and strictly decreasing in $(x_b, x_a + 2\pi)$.*

Proof. Assume there is an interval $I \subset I_1 \cup I_2$ in which P_f strictly increases. Choose $x_1 < x_2$ in I and note that, by Lemma 3,

$$\begin{vmatrix} f(x_1 + h) & f(x_1) \\ f(x_2 + h) & f(x_2) \end{vmatrix} < 0 \quad \text{and} \quad \begin{vmatrix} f(x_1) & f(x_1 - h) \\ f(x_2) & f(x_2 - h) \end{vmatrix} < 0$$

holds for small positive h. Since $f(x_a - h) > 0$ and $f(x_a + h) \geq 0$ we obtain

$$D_f\begin{pmatrix} x_1 & x_2 & x_3 \\ -h & 0 & y \end{pmatrix}$$
$$= \begin{vmatrix} f(x_1+h) & f(x_1) & f(x_1-h) \\ f(x_2+h) & f(x_2) & f(x_2-h) \\ f(x_a+h) & f(0) & f(x_a-h) \end{vmatrix}$$
$$= f(x_a+h)\begin{vmatrix} f(x_1) & f(x_1-h) \\ f(x_2) & f(x_2-h) \end{vmatrix} f(x_a-h)\begin{vmatrix} f(x_1+h) & f(x_1) \\ f(x_2+h) & f(x_2) \end{vmatrix}$$
$$< 0,$$

again a contradiction. The function $P_f = f'/f$ cannot be constant in any interval of the form $(x_a - \varepsilon, x_a)$ or $(x_b, x_b + \varepsilon)$ for small positive ε, because $f \neq 0$ as a solution of a differential equation $f' = Cf$, could not approach zero at x_a or x_b, respectively. This shows that there is an interval I suitable for the application of Lemma 8, and so f is continuously differentiable in x_M, and therefore in the whole of $(x_b, x_a + 2\pi)$. Lemma 9 then implies that f'/f is actually strictly decreasing in that interval. ∎

LEMMA 11. *Assume that f is zero-free. Then the assertions* (a), (b) *of Theorem* 3 *hold. Furthermore, P_f is strictly monotone in each of the intervals mentioned in Theorem* 3(c).

Proof. We have $x_m := x_a = x_b$, and P_f can be discontinuous only at x_M and/or x_m. Hence P_f has at most two discontinuities. Assume there is an interval where P_f is strictly monotone. Then by Lemma 8 we infer that P_f is continuous on at least one of the two possible points, and so has at most one discontinuity.

Hence, if P_f has two discontinuities, then it must be a step-function jumping from its minimum to its maximum and back, with the property

$$2P_f(x) = P_f(x+0) + P_f(x-0), \qquad x \in \mathbb{R}.$$

This completes the proof in this case.

If P_f decreases strictly somewhere (in an interval) and has a discontinuity, then that one must be at x_m. Then P_f must be decreasing in $(x_m, x_m + 2\pi)$, and the result follows. A similar argument works if the discontinuity is at x_M.

Now assume that P_f is continuous. Then, by Lemma 7(a), f' is periodically monotone and has therefore exactly two sign changes in a period. According to Lemma 9, P_f always moves strictly monotonically from its (positive) maximum to its (negative) minimum. Since it hits zero on each of these passages, there can be at most two such zeros in a period: P_f is periodically monotone.

The assertion about the intervals mentioned in (c) follows immediately from Lemma 9. ▮

We begin the discussion concerning second derivatives in the intervals where P_f is strictly monotone. For small $h > 0$ and $x_1 < x_2 < x_3 < x_1 + 2\pi$ we have

$$D_f\begin{pmatrix} x_1 & x_2 & x_3 \\ -h & 0 & h \end{pmatrix}$$
$$= \begin{vmatrix} (f(x_1+h)+f(x_1-h)-2f(x_1))/h^2 & (f(x_1)-f(x_1-h))/h & f(x_1) \\ (f(x_2+h)+f(x_2-h)-2f(x_2))/h^2 & (f(x_2)-f(x_2-h))/h & f(x_2) \\ (f(x_3+h)+f(x_3-h)-2f(x_3))/h^2 & (f(x_3)-f(x_3-h))/h & f(x_3) \end{vmatrix}.$$

In any point x where the limit exists, we define

$$f_s''(x) := \lim_{h\to 0} h^{-2}(f(x+h)+f(x-h)-2f(x)).$$

It is then clear, that

$$\begin{vmatrix} f_s''(x_1) & f'(x_1) & f(x_1) \\ f_s''(x_2) & f'(x_2) & f(x_2) \\ f_s''(x_3) & f'(x_3) & f(x_3) \end{vmatrix} < 0 \Rightarrow f \notin \mathrm{CPF}_3, \tag{25}$$

assuming that the $f_s''(x_j)$ exist. It is easily seen that in any interval in which $f''_\pm$ exists the relation

$$f_s''(x) = \tfrac{1}{2}(f''_+(x) + f''_-(x)) \tag{26}$$

holds.

LEMMA 12. *Let I be an open interval in which P_f is strictly monotone. Then $f \in C^2(I)$. Furthermore, f''/f is piece-wise monotone in I.*

Note that the same conclusion holds trivially in any interval where $P_f = $ const. or $f \equiv 0$.

Proof. We first assume that I contains neither of the points x_M, x_a, x_b. Then I is subset of either I_1 or I_2. Lemma 8 with (26) show that f_s'' exists in I, and that f_s''/f' is decreasing there. Once we have shown that f_s'' is continuous in I, then, because of the monotonicity, the same holds for $f''_\pm$.

for $f''_{\pm}$. And since f'' exists almost everywhere in I, this suffices to settle our claim.

Assume that f''_s is not continuous at $\bar{x} \in I$. Since f''_s/f' is decreasing, we have

$$\lim_{\varepsilon\to 0^+} \frac{f''_s(\bar{x}-\varepsilon)}{f'(\bar{x}-\varepsilon)} > \lim_{\varepsilon\to 0^+} \frac{f''_s(\bar{x}+\varepsilon)}{f'(\bar{x}+\varepsilon)}.$$

Assume $f'(\bar{x}) < 0$. Then we choose a point x_3 in I with $P_f(x_3) < P_f(\bar{x})$ so that

$$\begin{aligned}
&\lim_{\varepsilon\to 0^+} \begin{vmatrix} f''_s(\bar{x}-\varepsilon) & f'(\bar{x}-\varepsilon) & f(\bar{x}-\varepsilon) \\ f''_s(\bar{x}+\varepsilon) & f'(\bar{x}+\varepsilon) & f(\bar{x}+\varepsilon) \\ f''_s(x_3) & f'(x_3) & f(x_3) \end{vmatrix} \\
&= (f''_s(\bar{x}-0) - f''_s(\bar{x}+0)) \begin{vmatrix} f'(\bar{x}) & f(\bar{x}) \\ f'(x_3) & f(x_3) \end{vmatrix} \\
&= \left(\frac{f''_s(\bar{x}-0)}{f'(\bar{x}-0)} - \frac{f''_s(\bar{x}+0)}{f'(\bar{x}+0)} \right) \left(\frac{f'(\bar{x})}{f(\bar{x})} - \frac{f'(x_3)}{f'(x_3)} \right) f'(\bar{x}) f(\bar{x}) f(x_3) \\
&< 0.
\end{aligned}$$

In view of (25) this contradicts $f \in \mathrm{CPF}_3$. A similar method works if $f'(\bar{x}) > 0$.

Assume now that P_f is strictly decreasing in I. We show that f''/f cannot attain a local maximum in I, which implies the piece-wise monotonicity. If $x_1 < x_2 < x_3$ are three points in I with

$$\frac{f''(x_1)}{f(x_1)} = \frac{f''(x_3)}{f(x_3)} < \frac{f''(x_2)}{f(x_2)}$$

then

$$\begin{aligned}
&\begin{vmatrix} f''(x_1) & f'(x_1) & f(x_1) \\ f''(x_2) & f'(x_2) & f(x_2) \\ f''(x_3) & f'(x_3) & f(x_3) \end{vmatrix} \\
&= f(x_1) f(x_2) f(x_3) \left(\frac{f''(x_2)}{f(x_2)} - \frac{f''(x_3)}{f(x_3)} \right) \left(\frac{f'(x_3)}{f(x_3)} - \frac{f'(x_1)}{f(x_1)} \right) < 0,
\end{aligned}$$

another contradiction. If P_f is strictly increasing in I, then a similar argument can be used to complete the proof for these intervals I.

If I contains x_M then P_f strictly decreases in I, and we know already that (applying the previous part to the two subintervals of I which do not contain x_M) f'' exists and is continuous in $I \setminus \{x_M\}$. For some $\varepsilon > 0$ we see that $f''(x) \le 0$ in $J := (x_M - \varepsilon, x_M) \cup (x_M, x_M + \varepsilon)$ since f is concave in a neighborhood of its maximum at x_M. Furthermore, f''/f can be assumed to be monotonic in either of the two sub-intervals of J by what has already been established in the present proof.

We can also assume that f'' is bounded in J. In fact, otherwise there would be a sequence $\{x_k\} \subset J$ with $x_k \to x_M$, so that for $z_1, z_2 \in J$ with $x_M < z_1 < z_2$ and k large

$$\operatorname{sgn}\begin{vmatrix} f''(x_k) & f'(x_k) & f(x_k) \\ f''(z_1) & f'(z_1) & f(z_1) \\ f''(z_2) & f'(z_2) & f(z_2) \end{vmatrix} = \operatorname{sgn}\left(f''(x_k) \begin{vmatrix} f'(z_1) & f(z_1) \\ f'(z_2) & f(z_2) \end{vmatrix} \right) = -1$$

which is impossible.

The monotonicity and boundedness of f''/f in J implies the existence of the one-sided limits $f''(x_M \pm 0) = f''_{\pm}(x_M)$, and it remains to show that $f''_{+}(x_M) = f''_{-}(x_M)$. If, $f''_{+}(x_M) < f''_{-}(x_M)$, then we choose $z \in (x_M - \varepsilon, x_M)$ and obtain

$$\lim_{\delta \to 0^+} \begin{vmatrix} f''(z) & f'(z) & f(z) \\ f''(x_M - \delta) & f'(x_M - \delta) & f(x_M - \delta) \\ f''(x_M + \delta) & f'(x_M + \delta) & f(x_M + \delta) \end{vmatrix} < 0,$$

which again is impossible. A similar reasoning works for the case $f''_{+}(x_M) > f''_{-}(x_M)$, and so we are done.

Finally, if $x_a = x_b = x_m \in I$, and f has a positive minimum in x_m, a similar discussion can be applied. ∎

The final step in our proof of Theorems 2 and 3 is to show that $f \in L(I)$ for every interval I as described in Lemma 12. It clearly suffices to do so for closed subintervals $T := [x_1, x_2] \subset I$. We make use of the de-la-Vallée–Poussin kernels, using the smoothness properties just obtained.

The function $f_n := V_n * f$ belongs to CPF_3 for $n \in \mathbb{N}$, and we have

$$f'_n = (V_n * f)' = V_n * (f'), \qquad f''_n = (V_n * f)'' = V_n * (f'').$$

From (18) we obtain $f_n(x) \to f(x)$, $f'_n(x) \to f'(x)$, $f''_n(x) \to f''(x)$ in T, and the convergence is uniform in T. From our assumption we know that

$$\left| f''(x) f(x) - (f'(x))^2 \right| \ge \delta > 0, \qquad x \in T,$$

and therefore

$$\left|f_n''(x)f_n(x) - (f_n'(x))^2\right| \geq \delta/2 > 0, \qquad x \in T,$$

for large n. Using (8) we deduce that the functions $\log|f_n''(x)f_n(x) - (f_n'(x))^2|$ are concave in T, which therefore holds true for its limit $\log|f''(x)f(x) - (f'(x))^2|$ as well. Hence $f \in L(T)$.

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